# On the scientific contribution of Professor Moser

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1 Research and education in actuarial mathematics

*1.1 Overview of C. Moser's scientific work*

- C. Moser obtained the doctor title in mathematics and physics in 1887 in Bern

"Über Gebilde, welche durch Fixation einer spärischen Curve und Fortbewegung des Projectionscentrums entstehen"

He obtained the honorary doctorate in actuarial science from Lausanne at his 70th birthday

- 32 scientific publications in: mathematics (geometry), astronomy, geography, accident, health and life insurance

"Über eine mit der Umlaufszeit der Planeten zusammenhängende Relation", *Mitteilungen der Naturforschende in Gesellschaft in Bern*, 1899.

"Die Intensität der Sterblichkeit und die Intensitätsfunktion", *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker*, Heft 1, 1906.
Inaugural article
Since 2011 *European Actuarial Journal*

"Integralgleichungen und sich erneuernde Gesamtheiten", *Proceedings of the IX-th International Congress of Actuarial Science*, Stockholm, 1930.

- Several expertises in accident and health insurance, mostly published by Federal department of industry. (He worked from 1890 as insurance mathematician for the F.D.I.)

*1.2 Actuarial education at Universities*

- Actuarial studies began at end of 19-th century, in Vienna and Göttingen

- Professors of actuarial mathematics existed in Basel and Polytechnic school of Zurich at end of 19-th

- Seminar of insurance mathematics was created in Bern on December 30 1901, with active support of C. Moser

- “Notiz betreffend den Unterricht der Versicherungswissenschaft auf der Universität”, *Proceedings of the IV-th International Congress of Actuarial Science*, New York, 1903.

## 2 The Moserian renewal model

### *2.1 Introduction*

- Renewal problem has been major topic of insurance mathematicians during last 100 years

- "Beiträge zur Darstellung von Vorgängen und des Beharrungszustandes bei einer sich erneuerenden Gesamtheit", *M.V.S.V.*, Heft 21, 1926.

- Has led to many publications, many of them from swiss authors and in the M.V.S.V.

- Some previous contributions in this topic are:

Kinkelin (1869), Basel
Schaertlin (1899), *Zeitschr. f. Schweiz. Statistik*
Herbelot (1909), *Bull. Trim. de l'Inst. des Act. Fr.*
Risser (1912), *Bull. Trim. de l'Inst. des Act. Fr.*
Schenker (1918), *M.S.V.M.*
Alder (1923), Bern

- Much of Moser (1926) was part of his lectures at the University of Bern, since 1898

“Die Bedeutung der Integralgleichungen für die Mathematik der sozialen Versicherung ist erstmals eindeutig erfasst worden durch die Vorlesungen von Prof. Dr. Chr. Moser über das Gebiet der Erneuerungsfunktion und Integralgleichung.”
E. Zwinggi (1931, Bern)

*2.2 Illustration of the model*

• Evolution of various quantities of a collectivity (e.g. pension fund)

Values at stationary state of particular interest

• Two evolutionary schemes

I Departing members (death, disability, etc.) are not replaced

II Any departure is immediately replaced by a new one (of fixed entrance age, e.g. 30)

How to determine the intensity of new arrivals viz. the renewal function? How does it behave in steady state?

I The closed collectivity - “Die geschlossene Gesamtheit”

• Departures allowed, entrances not

$H$: initial number of members

$p(t)$: probability of maintaining membership after $t$ units of time; $p(0) = 1$, $p(\infty) = 0$, e.g. $p(t) = \mathrm{e}^{-t}$

$H(t) = Hp(t)$: number of members at time $t$

## II The renewable collectivity - “Die sich erneuende Gesmatheit”

- $H(t) = H$ constant (reasonable because interest in steady state)

$H\varphi(\tau)\mathrm{d}\tau$: number of renewals during $(\tau, \tau + \mathrm{d}\tau)$

$H\varphi(\tau)\mathrm{d}\tau p(t-\tau)$: previous quantity at time $t$, $t > \tau$

- Determination of the renewal function $\varphi$

$$H = \underbrace{Hp(t)}_{\text{remained members}} + \underbrace{\int_0^t H\varphi(\tau)\mathrm{d}\tau p(t-\tau)}_{\text{departed i.e. renewed members}} \iff$$

$$1 = p(t) + \int_0^t \varphi(\tau) \underbrace{p(t-\tau)}_{\text{kernel}} \mathrm{d}\tau \qquad (1)$$

Special Volterra integral equation of 1st kind (1913, “Leçons professées à la Faculté des sciences de Rome par Vito Volterra”)

- For $p(t) = \mathrm{e}^{-t}$, (1) is satisfied by $\varphi(t) = 1$

- $\varphi(t) \to \left( \underbrace{\int_0^\infty p(\tau) \mathrm{d}\tau}_{\text{mean membership time}} \right)^{-1}$, as $t \to \infty$ (under condition)

## Processes and their propagation - "Vorgänge und ihre Übertragung"

## I Closed collectivity

- $Hy(t)$: index of any related process at time $t$ (e.g. number of disabled, deaths, widows, days of illness); $y(\infty) = 0$

- $p(t) = 0 \nRightarrow Hy(t) = 0$

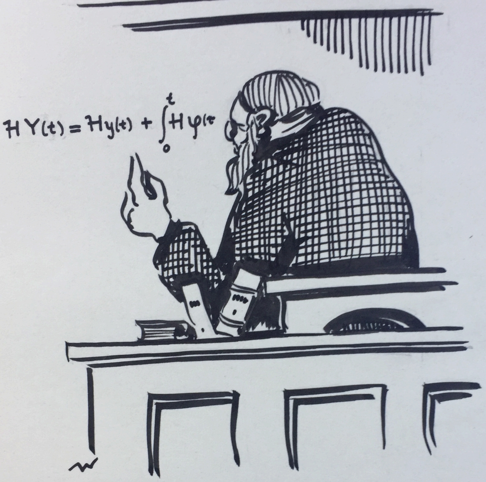
## II Renewable collectivity

- $Hy(t)$ observable and $\varphi(t)$ obtained from (1)
- $HY(t)$: renewable analogue of $Hy(t)$

$$HY(t) = \underbrace{Hy(t)}_{\text{from remained members}} + \underbrace{\int_0^t \underbrace{H\varphi(\tau)\mathrm{d}\tau}_{\text{number of renewals during } (\tau,\tau+d\tau)} \quad y(t-\tau)}_{\text{from departed i.e. renewed members}}$$

$$\Longleftrightarrow Y(t) = y(t) + \int_0^t \varphi(\tau) y(t-\tau)\mathrm{d}\tau \qquad (2)$$

Process of renewable collectivity $HY(t)$ is determined from process of closed collectivity $Hy(t)$




Moser legt der Schülerschar
Die Entwicklungsgleichung dar,
Eine Gleichung sehr apart
Nach Volterra, erster Art.

## Particular case

### I Closed collectivity

- $y(t) = -p'(t)$, probability density

$Hy(\tau)\mathrm{d}\tau$: number of departures during $(\tau, \tau + \mathrm{d}\tau)$

### II Renewable collectivity

- Departures during $(\tau, \tau + \mathrm{d}\tau)$ equals renewals during $(\tau, \tau + \mathrm{d}\tau)$

$\Leftrightarrow HY(\tau)\mathrm{d}\tau = H\varphi(\tau)\mathrm{d}\tau \Leftrightarrow Y(\tau) = \varphi(\tau)$

- (2) yields

$$\varphi(t) = -p'(t) + \int_0^t \varphi(\tau) \underbrace{\{-p'(t-\tau)\}}_{\text{kernel}} \mathrm{d}\tau,$$

Special Volterra integral equation of 2nd kind ($\varphi$ on the left)
It is the derivative of (1)

## Numerical illustration

- Departures by death only
- $y(t) = -p'(t)$
- Reserve - “Deckungskapital”

### I Closed collectivity

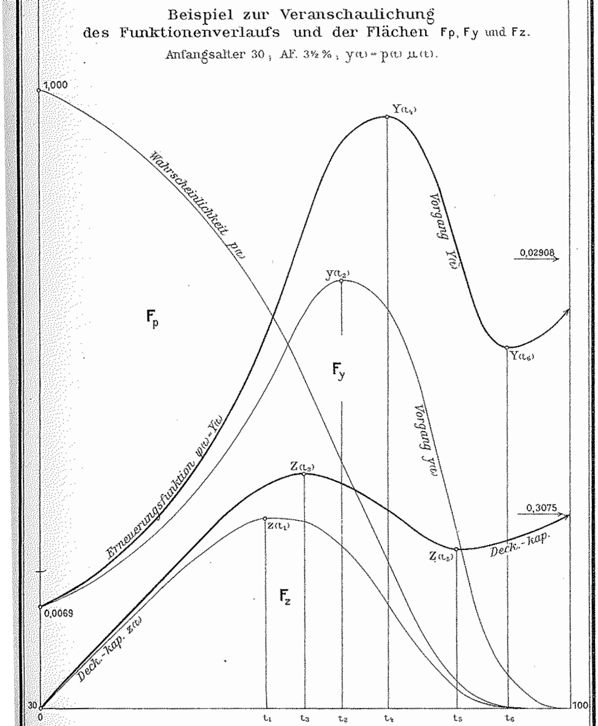
$Hz(t)$: reserve at time $t$
$z(t)$ depending on $p(\tau)$, for all $\tau \geq 0$, and on interest rate

### II Renewable collectivity

$HZ(t)$: renewable analogue of $Hz(t)$

$$Z(t) = z(t) + \int_0^t \varphi(\tau) z(t-\tau) \mathrm{d}\tau$$

Reserve of renewable collectivity $HZ(t)$ is determined from reserve of the closed collectivity $Hz(t)$




$H = 1$

$y(t)$ and $Y(t)$ shown 25 times larger

$F_p$ equals mean membership time

$F_y = 1$

- The dynamic of the reserve is strongly dependent on the dynamic of the population (compare $Z(t)$ and $Y(t)$)
- It was later shown that, without conditions, the renewal function does not always behave like a damped waveform

## *2.3 Generalizations and parallel reserach*

- Some generalizations which have directly followed are the following
- Stochastic generalizations

Richter (1941), *Math. Annalen*

Feller (1949), *Trans. of the Am. Math. Soc.*

Doob (1948), *Trans. of the Am. Math. Soc.*

Have lead to renewal processes, common in management and engineering

- Departures replaced by random number of arrivals

Bartlett

Kendall

Harris (1951), *Proc. of the Second Berkeley Sympos.*

Have lead to branching (or Galton-Watson) processes, common in biology and physics (e.g. reproduction of bacteria and propagation or particles)

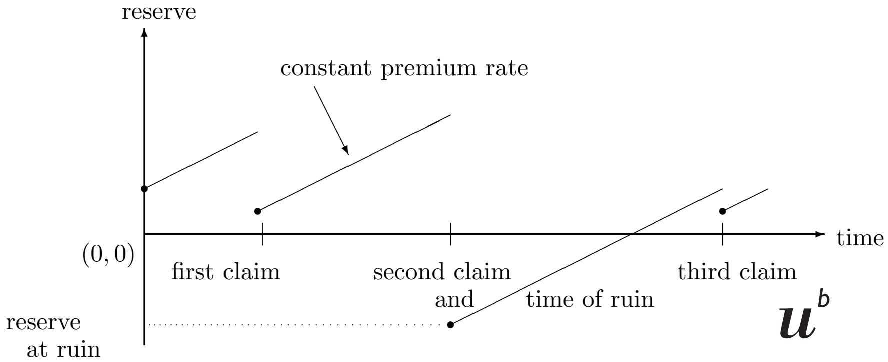
- See also: Ammeter (1955), *Vereinigung schweizerischer Versicherungsmathematiker 1905-1955*

- Some related parallel research is the following
- In mathematical ecology

Lotka

- In risk theory in Sweden

Lundberg (1903, 1926)
Cramer (1930)
Probability of ruin expressed in terms of a Volterra equation



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