

SOME SERIES EXPANSIONS

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Appendix of

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This appendix provides some complementary results that do not appear in the original publication.

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There may be other cases than those analyzed in Sections 3.1 and 3.2 where the integral of the m.g.f. M_1 given in (5) admits a closed-form expression. An alternative way of evaluating this m.g.f. is through power series expansions. This leads to a computable saddlepoint approximation under quite general step size distributions, with twice differentiable density. This appendix provides these series expansions and the resulting saddlepoint approximation, when the number of steps of random walk is fixed. Explicit expansions are provided for the gamma and the Weibull distributions.

Let $k \in \mathbb{N} \setminus \{0\}$, $l \in \mathbb{N}$ and $u > 0$ and define the functions

$$\psi_{k,l}(u) = \int_{(0,\infty)} x^{k-\frac{p}{2}} I_{\frac{p}{2}-1}(x) g^{(l)}\left(\frac{x}{u}\right) dl_1(x), \quad (42)$$

where g is the density of $R_1 = \|\mathbf{X}_1\|$ w.r.t. l_1 , assuming existence of its l -th derivative $g^{(l)}$. Then the differential equation

$$\psi'_{k,l}(u) = -\frac{\psi_{k+1,l+1}(u)}{u^2} \quad (43)$$

holds, assuming existence of $g^{(l+1)}$. Thus we have

$$\bar{M}_1(u) = 2^{\frac{p}{2}-1} \Gamma\left(\frac{p}{2}\right) \frac{\psi_{1,0}(u)}{u}.$$

From this, from (43) with $k = 1$ and $l = 0$ and from $\partial\|\mathbf{v}\|/\partial\mathbf{v} = \mathbf{z}$ follows

$$C_p(u) = \bar{K}'_1(u) = -\frac{1}{u} \left(1 + \frac{\psi_{2,1}(u)}{u\psi_{1,0}(u)}\right), \quad (44)$$

assuming existence of g' . According to (11), \bar{u} is the solution w.r.t. u of

$$-\frac{1}{u} \left(1 + \frac{\psi_{2,1}(u)}{u\psi_{1,0}(u)} \right) = \frac{r}{n}. \quad (45)$$

It follows from (43) and (44) that

$$C_p'(u) = \frac{1}{u^2} \left\{ 2 + \frac{\psi_{3,2}(u)}{u^2\psi_{1,0}(u)} - \left(1 - \frac{\psi_{2,1}(u)}{u\psi_{1,0}(u)} \right)^2 \right\},$$

which yields

$$\sigma_p^2(u) = \left(\frac{1}{u} \right)^{p+1} \left\{ - \left(1 + \frac{\psi_{2,1}(u)}{u\psi_{1,0}(u)} \right) \right\}^{p-1} \left\{ 2 + \frac{\psi_{3,2}(u)}{u^2\psi_{1,0}(u)} - \left(1 - \frac{\psi_{2,1}(u)}{u\psi_{1,0}(u)} \right)^2 \right\}, \quad (46)$$

assuming existence of g'' . From the replacement of the saddlepoint equation (45) into (46), one obtains

$$\sigma_p^2(\bar{u}) = \left(\frac{1}{\bar{u}} \right)^2 \left(\frac{r}{n} \right)^{p-1} \left(2 - \left(2 + \frac{\bar{u}r}{n} \right)^2 + \frac{\psi_{3,2}(\bar{u})}{\bar{u}^2\psi_{1,0}(\bar{u})} \right).$$

Proposition 4.1. *Let $k \in \mathbb{N} \setminus \{0\}$, $l \in \mathbb{N}$, $u > 0$, then the functions $\psi_{k,l}$ given by (42) admit the power series representation*

$$\psi_{k,l}(u) = \left(\frac{1}{2} \right)^{\frac{p}{2}-1} u^k \sum_{j=0}^{\infty} s_{j,k,l} \left(\frac{u}{2} \right)^{2j}, \quad (47)$$

where

$$s_{j,k,l} = \frac{m_{2j+k-1,l}}{j! \Gamma\left(\frac{p}{2} + j\right)} \quad (48)$$

and

$$m_{k,l} = \int_{(0,\infty)} x^k g^{(l)}(x) dl_1(x),$$

assuming existence of $g^{(l)}$.

Proof. The series expansions (47) can be justified by replacing $I_{p/2-1}$ in (42) by the ascending series

$$I_\nu(z) = \left(\frac{z}{2} \right)^\nu \sum_{j=0}^{\infty} \frac{\left(\frac{z}{2} \right)^{2j}}{j! \Gamma(\nu + j + 1)}, \quad \forall z \in \mathbb{C},$$

see e.g. 9.6.10 at p. 375 of Abramowitz and Stegun [26], and by integrating term by term.

These developments lead to the following general result.

Corollary 4.2 (Computable general saddlepoint approximation). *Assume g'' exists and let $r > 0$, then the saddlepoint \bar{u} at r , defined by (11), can be computed by*

$$\frac{\sum_{j=0}^{\infty} s_{j,2,1} \left(\frac{u}{2} \right)^{2j}}{\sum_{j=0}^{\infty} s_{j,1,0} \left(\frac{u}{2} \right)^{2j}} = - \left(1 + \frac{r}{n} u \right).$$

The m.g.f. of \mathbf{X}_1 as function of $u > 0$ can be computed by

$$\bar{M}_1(u) = \Gamma\left(\frac{p}{2}\right) \sum_{j=0}^{\infty} s_{j,1,0} \left(\frac{u}{2}\right)^{2j}. \quad (49)$$

The determinant of the Hessian matrix of K_1 at the saddlepoint \bar{u} can be computed by

$$\sigma_p^2(\bar{u}) = \left(\frac{1}{\bar{u}}\right)^2 \left(\frac{r}{n}\right)^{p-1} \left(2 - \left(2 + \frac{\bar{u}r}{n}\right)^2 + \frac{\sum_{j=0}^{\infty} s_{j,3,2} \left(\frac{\bar{u}}{2}\right)^{2j}}{\sum_{j=0}^{\infty} s_{j,1,0} \left(\frac{\bar{u}}{2}\right)^{2j}}\right). \quad (50)$$

The saddlepoint approximation to $q_{p,n}(r)$, the density of R_n , can be computed by (14) after replacing $\sigma_p^2(\bar{u})$ by (50) and $\bar{M}_1(u)$ by (49). The coefficients $s_{j,k,l}$ are given by (48) and their existence is assumed.

When applying Corollary 4.2 to specific step size distributions, we merely need the coefficients $s_{j,k,l}$ defined in (48). The next results provide these coefficients for two important generalizations of the exponential distribution: the gamma and the Weibull distributions. They provide sufficient flexibility to cope with many practical situations.

Proposition 4.3. *Assume gamma distributed lengths, i.e. $g(x) = \beta^\alpha/\Gamma(\alpha)x^{\alpha-1}e^{-\beta x}$, $\forall \alpha, \beta, x > 0$. For convenience, denote $s_{j,k,l}(\alpha) = s_{j,k,l}$. Then $\forall j \in \mathbb{N}$,*

$$s_{j,k,0}(\alpha) = \frac{\Gamma(\alpha + 2j + k - 1)}{\beta^{2j+k-1}\Gamma(\alpha)\Gamma\left(\frac{p}{2} + j\right)j!}, \quad \forall k \in \mathbb{N} \setminus \{0\}, \quad (51)$$

$$s_{j,k,1}(\alpha) = \beta\{s_{j,k,0}(\alpha - 1) - s_{j,k,0}(\alpha)\}, \quad \forall k \in \mathbb{N} \setminus \{0, 1\}, \quad (52)$$

and

$$s_{j,k,2}(\alpha) = \beta^2\{s_{j,k,0}(\alpha - 2) - 2s_{j,k,0}(\alpha - 1) + s_{j,k,0}(\alpha)\}, \quad \forall k \in \mathbb{N} \setminus \{0, 1, 2\}. \quad (53)$$

Thus (52) and (53) can be computed from (51).

Proof. The moment of order $k \in \mathbb{N}$ of the gamma distribution is $m_{k,0} = \Gamma(\alpha + k)/\{\beta^k\Gamma(\alpha)\}$. This formula and (48) yield (51).

Redenote $g = g_\alpha$, then it is easily seen that it satisfies the differential equation

$$g'_\alpha(x) = \beta\{g_{\alpha-1}(x) - g_\alpha(x)\}.$$

This result leads directly to (52).

Similarly, g_α satisfies the differential equation

$$g''_\alpha(x) = \beta^2\{g_{\alpha-2}(x) - 2g_{\alpha-1}(x) + g_\alpha(x)\}.$$

This result leads directly to (53).

Although Proposition 4.3 does hold for $\alpha = 1$, this case can be substantially simplified as follows.

Corollary 4.4. *Assume exponential lengths, i.e. $g(x) = \beta e^{-\beta x}$, $\forall \beta, x > 0$. Then $\forall j \in \mathbb{N}$,*

$$s_{j,1,0} = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\beta}\right)^{2j} \frac{\Gamma(\frac{1}{2} + j)}{\Gamma(\frac{p}{2} + j)}, \quad s_{j,2,0} = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\beta}\right)^{2j+1} \frac{\Gamma(\frac{3}{2} + j)}{\Gamma(\frac{p}{2} + j)} \quad (54)$$

$$s_{j,k,0} = \frac{\Gamma(2j+k)}{\beta^{2j+k-1} \Gamma(\frac{p}{2} + j) j!}, \quad \forall k \in \mathbb{N} \setminus \{0, 1, 2\}, \quad (55)$$

$$s_{j,k,1} = -\beta s_{j,k,0}, \quad \text{and} \quad s_{j,k,2} = \beta^2 s_{j,k,0}, \quad \forall k \in \mathbb{N} \setminus \{0\}. \quad (56)$$

Thus (56) can be computed from (54) and (55).

Proof. Both formulae in (54) are obtained from (51) simplified with the duplication formula

$$\Gamma(2z) = \pi^{-\frac{1}{2}} 2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right), \quad (57)$$

see e.g. 6.1.18 at p. 256 of Abramowitz and Stegun [26], whereas (55) is simply (51).

From $g'(x) = -\beta g(x)$ and $g''(x) = \beta^2 g(x)$ one obtains (56) directly.

Note the following closed-form expressions. When $p = 2$, (49) with (54) yield the power series of $\bar{M}_1(u) = \beta/\sqrt{\beta^2 - u^2}$: refer to the proof of Proposition 2.4 in Gatto [12]. When $p = 3$, the following result holds.

Corollary 4.5. *Assume exponential lengths, i.e. $g(x) = \beta e^{-\beta x}$, $\forall \beta, x > 0$, and $p = 3$. Then, $\forall u \in (-\beta, \beta)$,*

$$\sum_{j=0}^{\infty} s_{j,1,0} \left(\frac{u}{2}\right)^{2j} = \frac{\beta}{\sqrt{\pi u}} \log \frac{\beta + u}{\beta - u}, \quad \text{provided } u \neq 0, \quad (58)$$

$$\sum_{j=0}^{\infty} s_{j,2,1} \left(\frac{u}{2}\right)^{2j} = -\frac{2}{\sqrt{\pi}} \frac{\beta^2}{\beta^2 - u^2} \quad \text{and} \quad (59)$$

$$\sum_{j=0}^{\infty} s_{j,3,2} \left(\frac{u}{2}\right)^{2j} = \frac{4}{\sqrt{\pi}} \left(\frac{\beta^2}{\beta^2 - u^2}\right)^2. \quad (60)$$

Proof. The first equation in (54) with $p = 3$ leads to $\sum_{j=0}^{\infty} s_{j,1,0} (u/2)^{2j} = 2\pi^{-1/2} \sum_{j=0}^{\infty} 1/(1 + 2j)(u/\beta)^{2j}$. Then the series (41) leads to (58).

By using (54) and (56), one obtains $\sum_{j=0}^{\infty} s_{j,2,1} (u/2)^{2j} = -1/(2\pi^{1/2}) \sum_{j=0}^{\infty} (u/\beta)^{2j}$, which gives (59).

By using (55) and (56) we obtain $\sum_{j=0}^{\infty} s_{j,3,2} (u/2)^{2j} = \sum_{j=0}^{\infty} \Gamma(2[j + 3/2]) / \{\Gamma(j + 3/2)j!\} (u/\{2\beta\})^{2j}$. By using the duplication formula (57), one simplifies this last expression to $4\pi^{-1/2} \sum_{j=0}^{\infty} (j+1)(u/\beta)^{2j}$, which gives (60).

One can see that Corollary 4.5 is coherent with the results of Section 3.2.

Proposition 4.6. *Assume light-tailed Weibull distributed lengths, i.e. $g(x) = \alpha \beta x^{\alpha-1} \exp\{-\beta x^\alpha\}$, $\forall \alpha > 1, \beta > 0$ and $x > 0$. Then $\forall j \in \mathbb{N}$,*

$$s_{j,k,0} = \frac{\Gamma\left(1 + \frac{2j+k-1}{\alpha}\right)}{\beta^{\frac{2j+k-1}{\alpha}} \Gamma\left(\frac{p}{2} + j\right) j!}, \quad \forall k \in \mathbb{N} \setminus \{0\}, \quad (61)$$

$$s_{j,k,1} = (\alpha - 1)s_{j,k-1,0} - \alpha\beta s_{j,k+\alpha-1,0}, \quad \forall k \in \mathbb{N} \setminus \{0, 1\}, \quad (62)$$

and

$$s_{j,k,2} = (\alpha - 1)(\alpha - 2)s_{j,k-2,0} - 3(\alpha - 1)\alpha\beta s_{j,k+\alpha-2,0} + (\alpha\beta)^2 s_{j,k+2\alpha-2,0}, \quad \forall k \in \mathbb{N} \setminus \{0, 1, 2\}. \quad (63)$$

Thus (62) and (63) can be computed from (61).

Proof. Formula (61) follows from the moment of order k of the Weibull distribution, which is $m_{k,0} = \beta^{-k/\alpha} \Gamma(1 + k/\alpha)$, $\forall k \in \mathbb{N}$.

By differentiation one obtains

$$g'(x) = \{(\alpha - 1)x^{-1} - \alpha\beta x^{\alpha-1}\}g(x)$$

as well as

$$g''(x) = \{(\alpha - 1)(\alpha - 2)x^{-2} - 3(\alpha - 1)\alpha\beta x^{\alpha-2} + (\alpha\beta)^2 x^{2(\alpha-1)}\}g(x),$$

which lead to (62) and (63).

The exponential and Weibull distributions have the following relation: if X follows the exponential distribution with parameter β , given in Proposition 4.4, then $\forall \alpha > 0$, $X^{1/\alpha}$ follows the Weibull distribution, given in Proposition 4.6. The Weibull distribution is defined $\forall \alpha > 0$, but its m.g.f. exists around zero if $\alpha \geq 1$: the distribution is light-tailed when $\alpha \geq 1$. Theorem 4.6 is indeed restricted to the light-tailed Weibull distribution.